Math 53: Multivariable Calculus **Sections 212, 213**

Worksheet for 2020-09-04

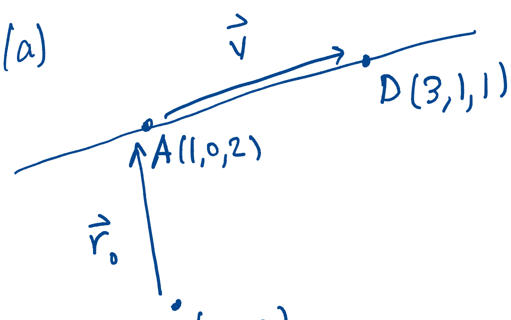
Problem 1. Let

$$A = (1, 0, 2)$$
$$B = (0, 3, 4)$$
$$C = (2, -2, 0)$$
$$D = (3, 1, 1)$$

(I picked these points randomly so there's no guarantee the arithmetic will be pretty.) Compute:

(a) An equation for the line through the points A, D.
(b) An equation for the line through the point C which is parallel to the line from (a).
(c) An equation for the plane containing the points A, B, C.
(d) An equation for the line through the point A which is perpendicular to the plane from (c).
(e) The area of the triangle with vertices A, B, C.
(f) The distance from C to the line from (a).
(g) The distance from D to the plane from (c).
(h) The angle (between 0 and $\pi/2$) formed between the line from (a) and the line through A and C. (Express your answer in terms of inverse trig functions, or use a calculator to get an approximate numerical answer.)
(i) The angle (between 0 and $\pi/2$) formed between the line from (a) and the plane from (c). (Ditto.)

(a)



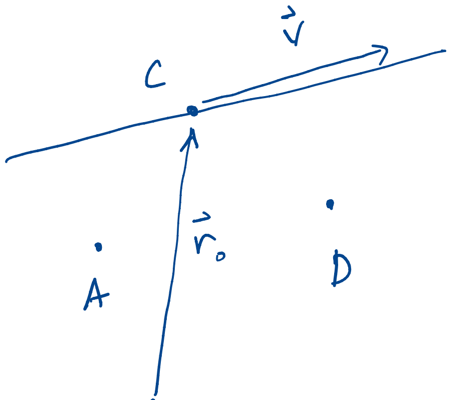
$\vec{r}_0 = \langle 1, 0, 2 \rangle$

$\vec{v} = \langle 2, 1, -1 \rangle$

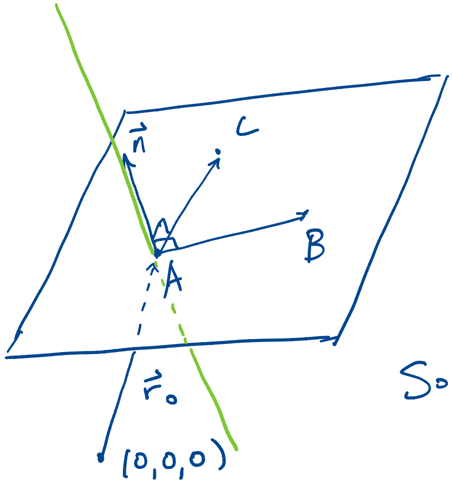
$\vec{r}(t) = \langle 1, 0, 2 \rangle + t \langle 2, 1, -1 \rangle$

or equivalently $x = 1 + 2t, \quad y = t, \quad z = 2 - t$

(b) Just use $\vec{r}_0 = \langle 2, -2, 0 \rangle$ instead, same $\vec{v}$.



(c)



Want $\vec{n}$ orthogonal to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, so take

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 3 & 2 \\ 1 & -2 & -2 \end{vmatrix} = \langle -2, 0, 1 \rangle$$

So then the plane eq is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

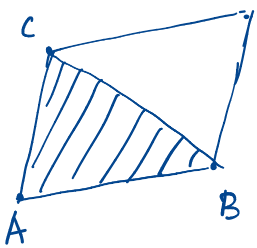
$$\langle -2, 0, 1 \rangle \cdot (\langle x, y, z \rangle - \langle 1, 0, 2 \rangle) = 0$$

or $\quad -2(x-1) + (z-2) = 0.$

(d) Indicated in green in above picture:

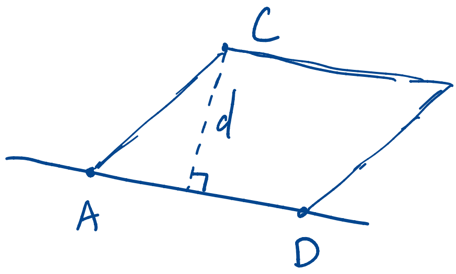
$$\vec{r}(t) = \langle 1, 0, 2 \rangle + t \langle -2, 0, 1 \rangle.$$

(e)



$$= \frac{1}{2} \underbrace{|\overrightarrow{AB} \times \overrightarrow{AC}|}_{\text{area of parallelogram}} = \frac{1}{2}\sqrt{5}.$$

(f)

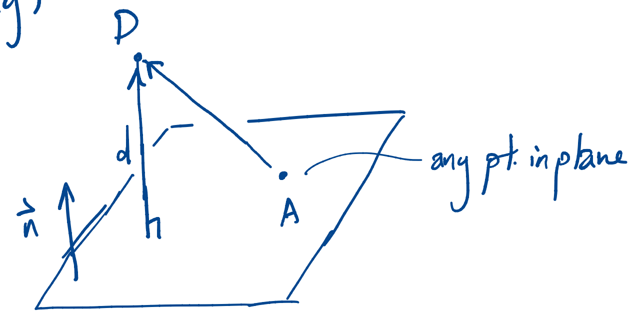


$$d = \frac{\text{area of parallelogram}}{|\overrightarrow{AD}|}$$

$$= \frac{|\overrightarrow{AC} \times \overrightarrow{AD}|}{|\overrightarrow{AD}|}$$

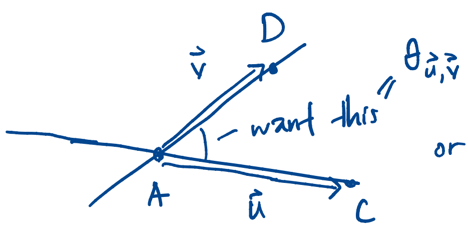
$$= \frac{|\langle 4, -3, 5\rangle|}{|\langle 2, 1, -1\rangle|} = \frac{5\sqrt{2}}{\sqrt{6}} = \frac{5}{\sqrt{3}} \text{ or } \frac{5\sqrt{3}}{3}$$

(g)

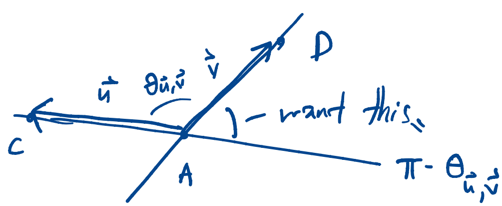


$$d = \left|\text{comp}_{\vec{n}}\, \overrightarrow{AD}\right| = \left|\frac{\langle 2, 1, -1\rangle \cdot \langle -2, 0, 1\rangle}{|\langle -2, 0, 1\rangle|}\right| = \sqrt{5}$$

(h) The situation is one of the following:



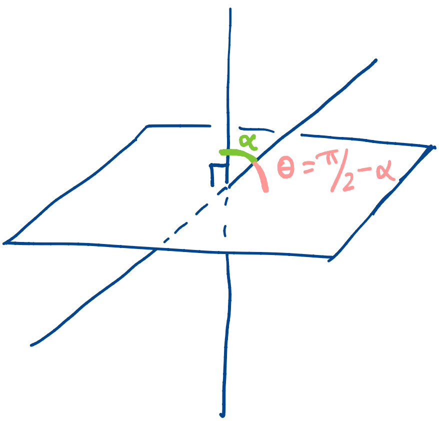
or maybe



Anyway, $\theta_{\vec{u},\vec{v}} = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \cos^{-1}\left(\frac{\langle 1,-2,-2\rangle\cdot\langle 2,1,-1\rangle}{3\sqrt{6}}\right)$

$\approx 74.21^\circ$ (already acute; i.e. this is the first picture above.)

(i) The desired angle is $\pi/2 -$ (angle between line and normal of plane), see below.



Use same method as (h) to compute α (make sure you get an answer between 0 and $\pi/2$).